PROPAGATION OF RANDOM VIBRATIONS IN A ROD WITH NONLINEAR PROPERTIES

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1. Let us consider arbitrary vibrations in a rod with nonlinear governing equation. The dynamic equations of the rod are:

$$Q' - mu^{\cdot\cdot} = 0, \qquad \varepsilon = u', \qquad Q = Q(\varepsilon, \varepsilon^{\cdot}) \tag{1.1}$$

where Q is the tensile force, ε the strain, u, the axial displacement, m the linear mass. Moreover, the prime denotes the derivative with respect to the coordinate x, and the dot, with respect to time. We moreover assume that the nonlinear function $Q(\varepsilon, \varepsilon^{\cdot})$ will be an odd function of its arguments.

Let us assume that the rod has the finite length l, where one of its ends $(x = 0)$ is free, and the other $(x = l)$ is loaded by a force p, which is a stationary random function of time with zero mathematical expectation.

The formulated problem will be the simplest of a series of problems originating in a study of the propagation of vibrations in such structures as aircraft, rockets, etc., because it is known that structural damping is nonlinear.

It is known that the formulated problem has no exact solution at present. An approximate solution, based on utilizing the method of statistical linearization [1–3], is proposed below.

In conformity with this method, the nonlinear third equation in (1.1) is replaced approximately by the linear equation

$$Q \approx h_1\varepsilon + h_2\varepsilon^{\cdot} \tag{1.2}$$

The linearization coefficients h_1 and h_2 are selected here from the condition that the linear relationship (1.2) would optimally approximate the original nonlinear relationship with respect to the criterion of minimum root-mean-square error. They are expressed as [2]:

$$h_1 = \frac{1}{\sigma_1^2}\iint\limits_{-\infty}^{\infty} Q(\varepsilon, \varepsilon^{\cdot})\,\varepsilon w(\varepsilon, \varepsilon^{\cdot})\,d\varepsilon\, d\varepsilon^{\cdot}, \qquad h_2 = \frac{1}{\sigma_2^2}\iint\limits_{-\infty}^{\infty} Q(\varepsilon, \varepsilon^{\cdot})\,\varepsilon^{\cdot} w(\varepsilon, \varepsilon^{\cdot})\,d\varepsilon\, d\varepsilon^{\cdot} \tag{1.3}$$

Here σ_1 and σ_2 are the mean-square values of the strain and its rate in the section x, and $w(\varepsilon, \varepsilon^{\cdot})$ is their joint probability density. Since the distribution law of ε and $\varepsilon^{\cdot}$ is unknown until the solution of the problem as a whole, it is assumed that it will be normal [2]

$$w(\varepsilon, \varepsilon^{\cdot}) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left(-\frac{\varepsilon^2}{2\sigma_1^2} - \frac{\varepsilon^{\cdot 2}}{2\sigma_2^2}\right) \tag{1.4}$$

It is seen from (1.3) and (1.4) that the linearization coefficients depend on the as yet unknown mean-square values of the strain and its rate $h_i = h_i(\sigma_1, \sigma_2)$.

Combining (1.2) and the first two equations of (1.1), we obtain an equation to determine the displacement

$$\frac{\partial}{\partial x}\left(h_1\frac{\partial u}{\partial x} + h_2\frac{\partial^2 u}{\partial x\,\partial t}\right) - m\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.5}$$

It agrees with the equation of longitudinal vibrations of a rod of Kelvin-Voigt material. It should, however, be kept in mind that its coefficients h_1 and h_2 depend on unknown characteristics of the solution $\sigma_1(x)$ and $\sigma_2(x)$.

2. Let us assign the spectral representation

$$p = \int_{-\infty}^{\infty} e^{i\omega t} V(\omega)\, d\omega \tag{2.1}$$

to a loading p acting at the end of the rod, where $V(\omega)$ is a random function of the type of white noise with intensity $S(\omega)$. The function $S(\omega)$ is called the spectral density of the loading.

By analogy with (2.1), we seek the solution of (1.5) as an integral canonical representation

$$u = \int_{-\infty}^{\infty} e^{i\omega t}\, \Phi(\omega, x) V(\omega)\, d\omega \tag{2.2}$$

Substituting (2.2) into (1.6), we obtain an equation for Φ

$$\frac{d}{dx}\left(C\frac{d\Phi}{dx}\right) + m\omega^2\Phi = 0, \qquad C = h_1 + i\omega h_2 \tag{2.3}$$

where C will be called below the complex stiffness, by analogy with the linear case.

Let us note that by virtue of (2.3), (1.5), the complex stiffness depends on σ_1 and σ_2, which will be unknown functions of x until the solution of the problem as a whole. The situation here is the same as in [4], where the method of harmonic linearization is utilized. It is impossible to construct an explicit solution of (2.3) for arbitrary $C(x)$. Hence, the case is examined below when σ_1 and σ_2, and in addition the complex stiffness C also, will be slowly varying functions of x. Then, by using the Steklov-Liouville method [5], an effective approximate solution of (2.3) can be constructed. In conformity with this method, let us make the change of variables

$$\Phi = C^{-1/4} v(y), \qquad y = \int_0^x (m/C)^{1/2}\, dx \tag{2.4}$$

We hence obtain in place of (2.3)

$$\frac{d^2v}{dy^2} + \left(\omega^2 - C^{-1/4}\frac{d^2}{dy^2} C^{1/4}\right) v = 0 \tag{2.5}$$

If the stiffness varies sufficiently slowly along the length of the rod, then the second member in the parentneses can be neglected. Equation (2.5) is then integrated easily, and the general expression is thereby found for Φ. The integration constants are determined from the boundary conditions. Integral canonical representations of the strain and its rate are easily obtained by means of the known integral canonical representation of u in (2.2)

$$\varepsilon = \int_{-\infty}^{\infty} e^{i\omega t}\, \Psi(\omega, x) V(\omega)\, d\omega, \qquad \varepsilon^{\cdot} = \int_{-\infty}^{\infty} i\omega e^{i\omega t}\, \Psi(\omega, x) V(\omega)\, d\omega \tag{2.6}$$

where

$$\Psi(\omega, x) = [C(x)]^{-3/4} [C(l)]^{-1/4} \frac{\sin \omega y(x)}{\sin \omega y(l)} \tag{2.7}$$

Using (2.6), we find the mean squares of ε and $\varepsilon^{\cdot}$

$$\sigma_1^2 = \int_{-\infty}^{\infty} |\Psi|^2 S(\omega)\, d\omega, \qquad \sigma_2^2 = \int_{-\infty}^{\infty} |\Psi|^2 S(\omega)\, d\omega \tag{2.8}$$

Since the unknown functions $\sigma_1(x)$ and $\sigma_2(x)$ enter the right and left sides of (2.8), these latter should be considered as equations to determine them. The unknown functions σ_1 and σ_2 are nonlinear on the right sides in the integrals with respect to the frequency and the coordinate x (integration with respect to x is included in y).

Therefore, (2.8) is a system of two nonlinear integral equations. After it has been solved, the interesting statistical characteristics of the vibration field is easily found by formulas presented above. Hence, the main problem is to solve the system of integral equations (2.8). In the most general case it can be solved by numerical methods, successive approximations say. Situations will be considered below for which an approximate analytical solution is possible.

3. The greatest simplicity of the system (2.8) is desirable in order to obtain analytical solutions. The expression for the square of the absolute value of Ψ is written in the notation

$$2y(x) = A(x) - i\omega B(x) \tag{3.1}$$

as follows:

$$|\Psi|^2 = [C(x)]^{-3/2}[C(l)]^{-1/2} \frac{\operatorname{ch} \omega^2 B(x) - \cos \omega A(x)}{\operatorname{ch} \omega^2 B(l) - \cos \omega A(l)} \tag{3.2}$$

The assumption of a slow change in the complex stiffness $C(x)$ has been introduced above. But it is clear from physical considerations that a slow change in σ_1 and σ_2, as well as in the complex stiffness which depends on them, is possible only for not too great damping. This means that the imaginary part in (2.3) should be small compared with the real part for substantial frequencies from the frequency range of the loading. The fact mentioned permits utilization of the following approximate formulas:

$$|C| = h_1, \quad A(x) = 2\int_0^x \left(\frac{m}{h_1}\right)^{1/2} dx, \quad B(x) = 2\int_0^x \left(\frac{m}{h_1}\right)^{1/2} \frac{h_2}{h_1}\, dx \tag{3.3}$$

which are valid to first order accuracy in the relatively small quantities.

The second simplification is more serious. The fact is that under slight damping the expression (3.2) is a rapidly varying function of the frequency with sharp peaks (these would be resonance peaks in the linear case). Hence, it is expedient to average $|\Psi|^2$ over a frequency band on the order of the spacing between the mentioned peaks, and then to integrate with respect to the frequency provided by (2.8). Performing this averaging as in [6], we obtain

$$|\Psi|_*^2 = h_{1l}^{-1/2} h_{1x}^{-3/2} \frac{\operatorname{ch} \omega^2 B(x)}{\operatorname{sh} \omega^2 B(l)} \tag{3.4}$$

where the subscript on the functions, here and henceforth, indicates the value of its argument so that h_{1x} is the value of h_1 at the section x, and h_{1l} at the section l.

Substituting (3.4) into (2.8), we obtain a simplified version of the system of integral equations

$$\sigma_1^2 = \int_{-\infty}^{\infty} |\Psi|_*^2 S(\omega)\, d\omega, \quad \sigma_2^2 = \int_{-\infty}^{\infty} |\Psi|_*^2 \omega^2 S(\omega)\, d\omega \tag{3.5}$$

It is understood that the averaging procedure described above assumes the existence of definite smoothness conditions in the loading spectral density.

Particular cases admitting of comparatively simple solutions are considered below.

4. Let the argument of the hyperbolic functions be small for the fundamental frequencies from the loading frequency range. Utilizing asymptotic formulas, we arrive at a system of equations from which it is seen directly that σ_1 and σ_2 are constant. The equations to determine them are:

$$\sigma_1^2 h_1^2 B(l) = \int_{-\infty}^{\infty} \frac{S(\omega)}{\omega^2}\, d\omega, \quad \sigma_2^2 h_1^2 B(l) = \int_{-\infty}^{\infty} S(\omega)\, d\omega \tag{4.1}$$

In the other limiting case, when the argument of the hyperbolic functions is large, utilization of the appropriate asymptotic formulas in (3.4) reduces the system (3.5) to the following:

$$\sigma_1^2 = \int_{-\infty}^{\infty} \frac{\exp \omega^2 [B(x) - B(l)]}{h_{1x}^{3/2} h_{1l}^{1/2}} S(\omega)\, d\omega, \qquad \sigma_2^2 = \int_{-\infty}^{\infty} \frac{\exp \omega^2 [B(x) - B(l)]}{h_{1x}^{3/2} h_{1l}^{1/2}} \omega^2 S\, d\omega \tag{4.2}$$

The system (4.2) is of value in itself. The analysis of a semi-infinite rod $x < l$ loaded by a random force at the section $x = l$ reduces exactly to this system. From the physical viewpoint, the transition to the system (4.2) corresponds to neglecting reflection of the vibrations from the free end of the rod.

The first difficulty to arise in solving the system (4.2) is the evaluation of the integrals with respect to the frequency. Such an evaluation is possible in general form for some classes of loading spectral densities. For example, if

$$S = D \mid \omega \mid^s e^{-\rho\omega^2}, \quad s > 0, \quad \rho > 0, \quad D > 0 \tag{4.3}$$

the situation reduces to tabulated integrals

$$\int_0^{\infty} z^{s-1} e^{-pz^2} dz = \frac{1}{2} p^{-1/2 s} \Gamma\left(\frac{s}{2}\right) \tag{4.4}$$

where Γ is the Euler Gamma function.

In this case the system (4.2) becomes after some simplification,

$$\left[\frac{\sigma_{1x}}{\sigma_{1l}} \left(\frac{h_{1x}}{h_{1l}}\right)^{3/4}\right]^{-4/(s+1)} = \left[\frac{\sigma_{2x}}{\sigma_{2l}} \left(\frac{h_{1x}}{h_{1l}}\right)^{3/4}\right]^{-4/(s+3)} = 1 - \frac{B(x) - B(l)}{\rho} \tag{4.5}$$

The left equality yields a direct connection between σ_{1x} and σ_{2x} while the differential equation

$$\frac{\partial}{\partial \sigma_{2x}} \left[\frac{\sigma_{2x}}{\sigma_{2l}} \left(\frac{h_{1x}}{h_{1l}}\right)^{3/4}\right]^{-4/(s+3)} \frac{d\sigma_{2x}}{dx} = -\frac{1}{\rho} \frac{dB(x)}{dx} \tag{4.6}$$

is easily obtained from the second.

Here, the unknown function of σ_{1x} and σ_{2x} is on the right side by virtue of the last formula in (3.3). The boundary values σ_{1l} and σ_{2l} should be found directly from the system (4.2) at $x = l$

$$\sigma_{1l} h_{1l} = \sigma_p, \qquad \sigma_{2l} h_{1l} = \sigma_{p'}. \tag{4.7}$$

where σ_p and $\sigma_{p'}$ are the mean-square values of the loading and its rate of change.

5. Let the rod material be linearly elastic with a power resistance law. Its governing equation is

$$Q = k\varepsilon + r \mid \varepsilon' \mid^{\mu} \operatorname{sign} \varepsilon' \tag{5.1}$$

where k, r, μ are positive constants.

The linearization coefficients and $B'(x)$ have the following expressions:

$$h_1 = k, \quad h_2 = d\sigma_2^{\mu-1}, \quad dB / dx = \gamma\sigma_2^{\mu-1} \tag{5.2}$$

$$d = \frac{r}{\sqrt{\pi}} 2^{\frac{1+\mu}{2}} \Gamma\left(1 + \frac{\mu}{2}\right), \quad \gamma = \left(\frac{m}{k}\right)^{1/2} \frac{d}{k}$$

In this case (4.6) becomes

$$\frac{4}{s+3} \left(\frac{\sigma_{2x}}{\sigma_{2l}}\right)^{-1-4/(s+3)} \frac{1}{\sigma_{2l}} \frac{d\sigma_{2x}}{dx} = \frac{\gamma}{\rho} \sigma_{2x}^{\mu-1} \tag{5.3}$$

which has the solution

$$\sigma_{2x} = \sigma_{2l} \left[1 + \frac{(s+3)\gamma\lambda}{4\rho} \sigma_{2l}^{\mu-1} (l - x)\right]^{-1/\lambda}, \quad \lambda = \mu - 1 + \frac{4}{s+3} \tag{5.4}$$

This expression allows some general deductions on the nature of the vibration field. For definiteness, let us speak of a semi-infinite rod $x < l$. It follows from (5.4) that for $\lambda > 0$ the vibration includes the whole rod, while for $\lambda < 0$ it is propagated only over the distance

$$L = (l - x)_* = \frac{4\rho}{(s+3)\gamma\,|\,\lambda\,|}\,\sigma_{2l}^{\,1-\mu} \tag{5.5}$$

The rest of the rod turns out to be fixed.

For $\lambda = 0$ the vibrations damp out exponentially.

The deductions obtained are valid approximately even for a finite rod, if only a substantial decrease in σ_2 occurs along its length l.

6. Unfortunately, the general case when the arguments of the hyperbolic functions in (3.4) are arbitrary cannot be investigated analytically in general form. The reason for this is the difficulty in integrating with respect to the frequency in (3.5).

However, this integration can be carried out in one particular case, albeit approximately. We speak of a narrowband high frequency loading. In this case the system (3.5) becomes

$$\sigma_1^2 = |\,\Psi\,(\Omega,\ x)\,|_*^{\,2}\sigma_p^2, \qquad \sigma_2^2 = |\Psi\,(\Omega,\ x)\,|_*^{\,2}\ \Omega^2\sigma_p^2 \tag{6.1}$$

where Ω is the mean loading frequency, and σ_p is its mean-square value. It is understood that the loading frequency band should be sufficiently wide so that the averaging with respect to the frequency carried out above would still be meaningful.

The system (6.1) can be rewritten as follows:

$$\sigma_2 = \Omega\sigma_1, \qquad \sigma_1^2 = |\,\Psi\,(\Omega,\ x)\,|_*^{\,2}\sigma_p^2 \tag{6.2}$$

so that we have the second equation of (6.2), in which σ_2 should be eliminated by utilizing the first equation, for the determination of σ_1.

The vibration field is analyzed comparatively simply on the basis of [6.2).

Homogeneous and strongly inhomogeneous fields can be investigated by the methods in Sect. 4. The intermediate case, the case of a slightly inhomogeneous field, has not generally been investigated successfully before. Let us start with it. It is hardly possible to obtain an exact solution of the integral equation (6.2) in this intermediate case. Hence, an approximate solution is indicated. It is known that if the nature of the solution of the equation can be conjectured, then application of direct methods yields satisfactory results. One of them, the method of collocations, is applied below.

Let us clarify the possible nature of the solution of (6.2). Since σ_1 is a mean-square quantity, it is inevitably positive or zero. If the vibration field is homogeneous, then $\sigma_1 = \text{const}$. If it is slightly inhomogeneous, then σ_1 changes somewhat along the length, decreasing as it recedes from the loaded end of the rod. It is hence evident that the dependence

$$\sigma_{1x} = \sigma_{1l}\, e^{-b\,(1-x/l)} \tag{6.3}$$

can approximate the vibration field sufficiently well if only the parameters σ_{1l} and b. are selected successfully. To determine these parameters we demand that (6.2) satisfy (6.3) in just two points, the ends of the rods. We obtain

$$\sigma_{1l}^2 = |\,\Psi\,(\Omega,\ l)\,|_*^{\,2}\sigma_p^2, \qquad \sigma_{10}^2 = |\,\Psi\,(\Omega,\ 0)\,|_*^{\,2}\sigma_p^2 \tag{6.4}$$

Since σ_{1l} is expressed by (6.3), then (6.4) is a system of algebraic or transcendental equations to determine σ_{1l} and b.

It is perfectly clear that the approximate solution thus constructed describes a homogeneous and slightly inhomogeneous vibration field well. The approximation (6.3) transmits the details of a slightly inhomogeneous field poorly. But the method of Sect. 4 yields good results precisely in this case. The situation reduces then to the integral equation

$$\sigma_{1l}^{\,2} = \sigma_p^2\, h_{1l}^{-1/2}\, h_{1x}^{-3/2} \exp \Omega^2\, [B\,(x) - B\,(l)] \tag{6.5}$$

Taking the logarithmic derivative of the right and left sides, we reduce this equation to the differential equation

$$\left[\frac{\partial}{\partial\sigma_{1x}}\ \ln\left(\sigma_{1x}{}^{e} h_{1x}{}^{3/2}\right)\right]\frac{d\sigma_{1x}}{dx}=\Omega^2\,\frac{dB\,(x)}{dx} \tag{6.6}$$

In evaluating the derivative with respect to σ_{1x} it should be kept in mind that σ_{2x} depends on σ_{1x} in conformity with the first formula in (6. 2).

The first of conditions (4. 7) will be the boundary condition for (6. 6) because it is valid for any kind of spectral density of the loading.

Therefore, the whole range of possible values of inhomogeneity in the vibration field is covered.

7. Let us consider an example. Let us speak of a material with the rheological equation (5. 1). Then the system (6. 4) is written thus:

$$\begin{aligned}\left(\frac{k\sigma_{1l}}{\sigma_p}\right)^2 &= \operatorname{cth}\left\{\gamma l\Omega^{\mu+1}\,\sigma_{1l}^{\mu-1}\,\frac{1-e^{-b\,(\mu-1)}}{b\,(\mu-1)}\right\}\\ \left(\frac{k\sigma_{1l}\,e^{-b}}{\sigma_p}\right)^2 &= \operatorname{sh}^{-1}\left\{\gamma l\Omega^{\mu+1}\sigma_{1l}^{\mu-1}\,\frac{1-e^{-b\,(\mu-1)}}{b\,(\mu-1)}\right\}\end{aligned} \tag{7.1}$$

It can be converted to the simpler form

$$R=(1-e^{-4b})^{-1/4},\qquad e^{2b}=\operatorname{ch}\left\{(BR)^{\mu-1}\,\frac{1-e^{-b\,(\mu-1)}}{b\,(\mu-1)}\right\} \tag{7.2}$$

if the notation

$$R=\frac{k\sigma_{1l}}{\sigma_p},\qquad B=\frac{\sigma_p}{k}\,(\gamma l\Omega^{\mu+1})^{1/(\mu-1)} \tag{7.3}$$

is introduced.

The dependences $R = R\ (b)$ and $B = B\ (b)$ constructed by means of (7. 2) are shown in Figs. 1 and 2 (these are the lines marked with the number 1).

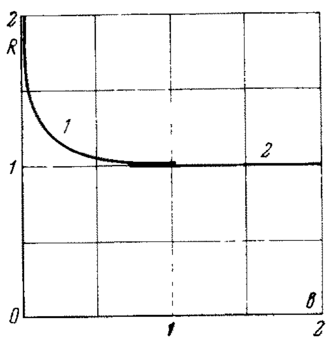

Fig. 1

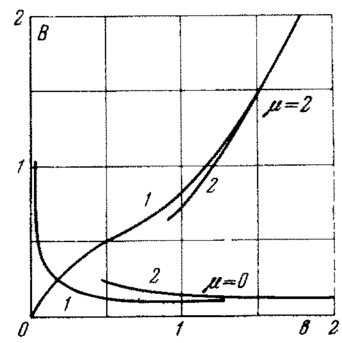

Fig. 2

Furthermore, the differential equation of an inhomogeneous field (6. 6) takes the form in this case

$$\frac{d\sigma_{1x}}{dx}=\gamma\,\frac{\Omega^{\mu+1}}{2}\,\sigma_{1x}{}^{\mu} \tag{7.4}$$

which has the solution

$$\sigma_{1x}=\sigma_{1l}\left[1+\frac{\mu-1}{2}\,\gamma\,\Omega^{\mu+1}\sigma_{1l}^{\mu-1}\,(l-x)\right]^{-1/(\mu-1)} \tag{7.5}$$

where σ_{1l} is determined from the boundary condition (4. 7).

This expression is suitable only for those x for which the right side remains real and positive. For $\mu > 1$ it is suitable for all x. If $\mu < 1$ it vanishes for a specific $x = x_*$

and thereafter is meaningless. Hence, for $< x_*$ we should take $\sigma_{1x} = 0$ since this solution satisfies (7.4) and continuously adjoins the solution (7.5) for the domain $x > x_*$.

To construct the solution (7.5) in Figs. 1, 2, let us first note that from boundary condition (4.7) it follows that:

$$R = 1 \tag{7.6}$$

Furthermore, we take the logarithm of the ratio between the mean-square strains at the point of load application ($x = l$) and at the free end ($x = 0$) as the degree of inhomogeneity b Such a definition of the index of inhomogeneity agrees with the approximation (6.3). In conformity with (7.5), we have in this case

$$b = \frac{1}{\mu - 1} \ln\left[1 + \frac{\mu - 1}{2} \gamma \Omega^{\mu+1} \sigma_{1l}^{\mu-1} l\right] \tag{7.7}$$

Utilizing the notation (7.3), we write this relationship thus:

$$B = \frac{1}{R}\left\{\frac{2}{\mu - 1}\left[e^{b(\mu-1)} - 1\right]\right\}^{1/(\mu-1)} \tag{7.8}$$

Presented in Figs. 1 and 2 are the dependences $R = R(b)$ and $B = B(b)$ constructed by means of (7.6) and (7.8), for two characteristic μ (these are the lines marked with the number 2). It is seen from (7.8) that for $\mu < 1$ the almost constant value

$$B = \left(\frac{2}{1 - \mu}\right)^{1/(\mu-1)} \tag{7.9}$$

corresponds to large b .

But B is proportional to the loading σ_p. What will it be for a loading less than that which corresponds to the limit (7.9) ? By virtue of (7.3), (7.6), (7.9), this will be the value for which the inequality

$$\tfrac{1}{2}(1 - \mu)\gamma l \Omega^{\mu+1} \sigma_{1l}^{\mu-1} > 1 \tag{7.10}$$

is satisfied.

But it then follows from (7.5) that the vibration does not reach the free end, i.e. we have $\sigma_{10} = 0$. Therefore, an infinite index of inhomogeneity corresponds to this case.

As has already been mentioned above, in the case of a slightly inhomogeneous vibration field ($b < 1$, say), it is necessary to use line 1, while in the case of strongly inhomogeneous vibration ($b > 1$, say), line 2. It should be noted that lines 1 and 2 go over into each other sufficiently well. Therefore, we have provided for the whole range of possible values of b.

Finally, let us note that the line $\mu = 2$ will be characteristic for the family of lines $\mu > 1$, while the line $\mu = 0$ is characteristic for the family $\mu < 1$. Taking account of this fact the following deductions can be made.

Firstly, a single value of b, and therefore, a single value of the statistical characteristics of the vibration field, corresponds to each value σ_p.

Secondly, for $\mu > 1$ an increase in σ_p results in a growth of the index of inhomogeneity of the vibration field, while for $\mu < 1$ it diminishes.

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ON THE STABILITY OF A NONAUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM

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Stability of the equilibrium position of a nonautonomous Hamiltonian system with two degrees of freedom is investigatied for the resonant case. The conditions of instability as well as those of formal stability are obtained.

1. We assume that the coordinate origin $q_i = p_i = 0$ corresponds to the position of equilibrium of the canonical system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1,2) \tag{1.1}$$

where H is a Hamiltonian function 2π-periodic in t and analytic in the vicinity of the point $q_i = p_i = 0$.

Let the linearized system be stable and all its multipliers be distinct. We assume that the Hamiltonian in (1.1) is transformed into

$$H = {}^1/_2\, \lambda_1 (q_1^2 + p_1^2) + {}^1/_2\, \lambda_2 (q_2^2 + p_2^2) + \sum_{\nu=3}^{\infty} h_{\nu_1\nu_2\nu_3\nu_4}(t)\, q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4} \tag{1.2}$$

by means of a real linear 2π-periodic canonical transformation [1]. In (1.2) $\pm i\lambda_1$ and $\pm i\lambda_2$ are the characteristic indices of the linearized system and ν_i are nonnegative integers

$$\nu = \nu_1 + \nu_2 + \nu_3 + \nu_4, \qquad h_{\nu_1\nu_2\nu_3\nu_4}(t + 2\pi) = h_{\nu_1\nu_2\nu_3\nu_4}(t)$$

We also assume that the condition

$$k_1\lambda_1 + k_2\lambda_2 \not\equiv 0 \qquad (\mathrm{mod}\ 1) \tag{1.3}$$

holds for the integers k_1 and k_2 satisfying the equalities $|k_1| + |k_2| = 3$ or $|k_1| + |k_2| = 4$. Then there exists [2] an analytic canonical transformation 2π-periodic in t, reducing the Hamiltonian (1.2) to the form

$$H = \lambda_1 r_1 + \lambda_2 r_2 + l_{2020} r_1^2 + l_{1111} r_1 r_2 + l_{0202} r_2^2 + O(|q|^5) \tag{1.4}$$

$$(|q| = \sqrt{q_1^2 + q_2^2 + p_1^2 + p_2^2}, \quad 2r_i = q_i^2 + p_i^2)$$

Coefficients $l_{\nu_1\nu_2\nu_3\nu_4}$ in (1.4) are independent of t. Let the quadratic form

$$l_{2020} r_1^2 + l_{1111} r_1 r_2 + l_{0202} r_2^2$$

be sign definite in the quadrant $r_1 \geqslant 0,\ r_2 \geqslant 0$. Then the position of equilibrium is